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INVARIANT SUBMODELS OF RANK TWO OF THE EQUATIONS OF GAS DYNAMICS

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Invariant submodels of rank two of systems of gas-dynamic equations with a general equation of state are described. All submodels (26 representatives) are divided into two classes — evolutionary and stationary. New relations and independent variables and the coefficients and right sides of the corresponding systems of equations are given.

As is known [1], a system of gas-dynamic equations with a general equation of state admits the 11-parameter Lie group. The basis of the corresponding Lie algebra is formed by the operators

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z, \quad X_4 = t\partial_x + \partial_u, \quad X_5 = t\partial_y + \partial_v, \quad X_6 = t\partial_z + \partial_w,$$
$$X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v, \quad X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w,$$
$$X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u, \quad X_{10} = \partial_t, \quad X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z.$$

The optimal system of subalgebras is constructed and given in [2, Table 6].

In the present paper, the invariant submodels of rank two of systems of gas-dynamic equations are analyzed within the framework of the SUBMODEL program. All such submodels (27 representatives) are obtained for two-parameter subgroups that correspond to the subalgebras $L_{2,l}$ from Table 6 of [2] (below, the numbering from Table 6 is used in references to the corresponding submodel).

Following [3], all the submodels considered (except for the partially invariant submodel 2.26) are reduced to one of the two systems. A system of equations of the evolutionary type (type E) has the form

$$U_t + UU_\xi + \frac{b_1}{R}P_\xi = a_1, \qquad V_t + UV_\xi = a_2, \qquad W_t + UW_\xi = a_3,$$
$$R_t + UR_\xi + RU_\xi = Ra_4, \quad P_t + UP_\xi + A(R,P)U_\xi = A(R,P)a_4,$$

where $b_1 = b_1(t,\xi)$, the functions a_i do not contain derivatives of the required functions, $A = Rc^2$, and $c = c(R,P)$. The characteristics are given by the equality $(d\xi - U dt)^3(Rd\xi^2 - 2RU d\xi dt + (RU^2 - Ab_1)dt^2) = 0$. According to the sense of the problem, $A > 0$ and $R > 0$, and it turns out that $b_1 > 0$. Therefore, all the characteristics are real.

A system of equations of the stationary type (type S) has the form

$$UU_\xi + VU_\eta + \frac{1}{R}b_{11}P_\xi = a_1, \qquad UV_\xi + VV_\eta + \frac{1}{R}b_{22}P_\eta = a_2,$$
$$UW_\xi + VW_\eta + \frac{1}{R}(b_{31}P_\xi + b_{32}P_\eta) = a_3, \qquad UR_\xi + VR_\eta + R(U_\xi + V_\eta) = Ra_4,$$
$$UP_\xi + VP_\eta + A(R,P)(U_\xi + V_\eta) = A(R,P)a_4,$$

where $b_{ij} = b_{ij}(\xi,\eta)$, and the functions a_i do not contain derivatives of the required functions.

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TABLE 1

Submodel	Coordinate system	Subalgebra basis	ξ
2.8	C	$X_4,\ X_7$	r
2.9	C	$X_1,\ \beta X_4 + X_7$	r
2.10	C	$X_4,\ X_1 + X_7$	r
2.20	D	$\alpha X_1 + \sigma X_3 + X_5,\ \beta X_1 + \tau X_2 + X_6,$ $\alpha^2 + \beta^2 + (\sigma + \tau)^2 = 1$	$x + \frac{\beta\sigma - \alpha t}{t^2 - \sigma\tau} y + \frac{\alpha\tau - \beta t}{t^2 - \sigma\tau} z$
2.21	D	$X_3 + X_5,\ X_2 - X_6$	x
2.22	D	$X_5,\ X_6$	x
2.23	D	$\alpha X_1 + X_2,\ X_3 + X_4$	$x - \alpha y - tz$
2.24	D	$\alpha X_1 + X_2,\ X_4$	z
2.25	D	$X_1,\ X_3 + X_4$	y
2.27	D	$X_2,\ X_3$	x

TABLE 2

Submodel	U	V	W
2.8	v	$u - x/t$	w
2.9	v	$u - \beta\theta$	w
2.10	v	$u + (\theta - x)/t$	w
2.20	$u + \frac{\beta\sigma - \alpha t}{t^2 - \sigma\tau} v + \frac{\alpha\tau - \beta t}{t^2 - \sigma\tau} w +$ $+ \frac{(\beta\sigma - \alpha t)(\tau z - ty) + (\alpha\tau - \beta t)(\sigma y - tz)}{(t^2 - \sigma\tau)^2}$	$v + \frac{\alpha t - \beta\sigma}{t^2 - \sigma\tau} u + \frac{\tau z - ty}{t^2 - \sigma\tau}$	$w + \frac{\beta t - \alpha\tau}{t^2 - \sigma\tau} u +$ $+ \frac{\sigma y - tz}{t^2 - \sigma\tau}$
2.21	u	$v + tw - z$	$tv - w - y$
2.22	u	$v - y/t$	$w - z/t$
2.23	$u - \alpha v - tw - z$	$\alpha u + v - \alpha z$	$tu + w - tz$
2.24	w	$u - x/t + \alpha(y/t)$	v
2.25	v	$u - z$	w
2.27	u	v	w

The characteristics are defined by the equality

$$(V d\xi - U d\eta)^3((RU^2 - Ab_{11})d\eta^2 - 2RUV d\xi d\eta + (RV^2 - Ab_{22})d\xi^2) = 0.$$

The first factor specifies three real families of characteristics. The second factor gives real characteristics if $\delta = b_{22}U^2 + b_{11}V^2 - b_{11}b_{22}c^2 \geqslant 0$, and it gives complex characteristics if $\delta < 0$.

Systems of equations of the evolutionary type are obtained from submodels 2.8–2.10, 2.20–2.25, and 2.27, and systems of equations of the stationary type are obtained from submodels 2.1–2.7 and 2.11–2.19.

The corresponding dependent and independent variables and the coefficients and right sides of the resultant systems of differential equations for all submodels are given in Tables 1–6 (R and P are functions of ρ and p in the new variables). The initial system of gas-dynamic equations is written in Cartesian coordinates (the coordinate system D) or in cylindrical coordinates (the coordinate system C).

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TABLE 3

Submodel	b_1	a_1	a_2	a_3	a_4
2.8	1	$(1/\xi)W^2$	$-(1/t)V$	$-(1/\xi)UW$	$-\left(\frac{1}{\xi}U+\frac{1}{t}\right)$
2.9	1	$(1/\xi)W^2$	$-(\beta/\xi)W$	$-(1/\xi)UW$	$-(1/\xi)U$
2.10	1	$(1/\xi)W^2$	$(W-\xi V)/t\xi$	$-(1/\xi)UW$	$-\left(\frac{1}{\xi}U+\frac{1}{t}\right)$
2.20	$1+\left(\frac{\beta\sigma-\alpha t}{t^2-\sigma\tau}\right)^2+\left(\frac{\alpha\tau-\beta t}{t^2-\sigma\tau}\right)^2$	$\frac{\sigma(\alpha\tau-\beta t)}{(t^2-\sigma\tau)^2}V+\frac{t(\alpha t-\beta\sigma)}{t^2-\sigma\tau}V+$ $+\frac{\tau(\alpha\tau-\beta t)}{(t^2-\sigma\tau)^2}W+\frac{t(\beta t-\alpha\tau)}{(t^2-\sigma\tau)^2}W$	$-\frac{t}{t^2-\sigma\tau}V+\frac{\tau}{t^2-\sigma\tau}W$	$\frac{\sigma}{t^2-\sigma\tau}V-\frac{t}{t^2-\sigma\tau}W$	$-\frac{2t}{t^2-\sigma\tau}$
2.21	1	0	0	0	$-2t/(1+t^2)$
2.22	1	0	$-(1/t)V$	$-(1/t)W$	$-2/t$
2.23	$1+\alpha^2+t^2$	$2\frac{tU+\alpha tV}{1+\alpha^2+t^2}-2\frac{(1+\alpha^2)W}{1+\alpha^2+t^2}$	$\alpha\frac{tU+\alpha tV}{1+\alpha^2+t^2}-\alpha\frac{(1+\alpha^2)W}{1+\alpha^2+t^2}$	$\frac{(1+t^2)U}{1+\alpha^2+t^2}+\frac{\alpha(1+t^2)V}{1+\alpha^2+t^2}-$ $-\alpha^2tW/(1+\alpha^2+t^2)$	0
2.24	1	0	$-(1/t)V+(\alpha/t)W$	0	$-1/t$
2.25	1	0	$-W$	0	0
2.27	1	0	0	0	0

TABLE 4

Submodel	Coordinate system	Subalgebra basis	ξ	η	U
2.1	C	$X_{10},\ X_7+\alpha X_{11},\ \alpha\neq 0$	r/x	$e^{-2\alpha\theta}(x^2+r^2)$	$e^{\alpha\theta}(xv-ru)/x^2$
2.2	C	$\alpha X_4+X_7,\ \beta X_4+X_{11}$	r/t	$x-\alpha\theta-\beta\ln\|t\|$	$v-r/t$
2.3	C	$X_4, X_7+\alpha X_{11},\ \alpha\neq 0$	$te^{-\alpha\theta}$	r/t	$1-(\alpha t/r)w$
2.4	C	$X_1,\ \beta X_4+X_7+\alpha X_{11},\ \alpha\neq 0$	r/t	$\theta-(1/\alpha)\ln\|t\|$	$v-r/t$
2.5	C	$X_7,\ X_{10}$	x	r	u
2.6	C	$X_1+X_7,\ X_{10}$	r	$x-\theta$	v
2.7	C	$\alpha X_1+X_7,\ X_4+X_{10}$	r	$x-(1/2)t^2-\alpha\theta$	v
2.11	C	$X_1,\ \beta X_4+X_7+X_{10}$	$\theta-t$	r	$w/r-1$
2.12	D	$X_{10},\ X_{11}$	z/y	$(y^2+z^2)/x^2$	$(x/y^2)(yw-zv)$
2.13	D	$X_4,\ X_{11}$	y/t	z/t	$v-y/t$
2.14	D	$X_4,\ \alpha X_5+X_{11},\ \alpha\neq 0$	$y/t-\alpha\ln\|t\|$	z/t	$v-y/t-\alpha$
2.15	D	$X_1,\ \beta X_4+\alpha X_5+X_{11},\ \alpha\neq 0$	$y/t-\alpha\ln\|t\|$	z/t	$v-y/t-\alpha$
2.16	D	$X_1,\ \beta X_4+X_{11}$	y/t	z/t	$v-y/t$
2.17	D	$X_1,\ X_{10}$	y	z	v
2.18	D	$X_3,\ X_4+\alpha X_6+X_{10}$	$x-t^2/2$	y	$u-t$
2.19	D	$X_1,\ X_4+X_{10}$	y	z	v

TABLE 5

Submodel	V	W	b_{11}	b_{22}	b_{31}
2.1	$(2/r)e^{-\alpha\theta}(xru+r^2v-\alpha(x^2+r^2)w)$	w	$(1+\xi^2)^2/\eta$	$4\eta(1+\alpha^2(1+1/\xi^2))$	0
2.2	$u-x/t-(\alpha t/r)w-\beta$	w	1	$1+\alpha^2/\xi^2$	0
2.3	$(1/t)(v-r/t)e^{\alpha\theta}$	$u-x/t$	α^2/η^2	$1/\xi^2$	0
2.4	$(t/r)w-1/\alpha$	$u-\beta\theta$	1	$1/\xi^2$	0
2.5	v	w	1	1	0
2.6	$u-(1/r)w$	w	1	$1+1/\xi^2$	0
2.7	$u-t-(\alpha/r)w$	w	1	$1+\alpha^2/\xi^2$	0
2.11	v	$u-\beta t$	$1/\eta^2$	1	0
2.12	$(2/x^2)(xyv+xzw-(y^2+z^2)u)$	u	$(1+\xi^2)^2/\eta$	$4\eta(1+\eta)$	0
2.13	$w-z/t$	$u-x/t$	1	1	0
2.14	$w-z/t$	$u-x/t$	1	1	0
2.15	$w-z/t$	$u-\beta\ln\|t\|$	1	1	0
2.16	$w-z/t$	$u-\beta\ln\|t\|$	1	1	0
2.17	w	u	1	1	0
2.18	v	$w-\alpha t$	1	1	0
2.19	w	$u-t$	1	1	0

TABLE 6

Submodel	b_{32}	a_1	a_2	a_3	a_4
2.1	$-2\alpha(\sqrt{(1+\xi^2)\eta}/\xi)$	$\frac{2\xi}{1+\xi}U^2 - \frac{1}{\eta}UV - UW$	$2\frac{\eta}{(1+\xi^2)^2}U^2 + \frac{1}{2\eta}V^2 - VW$	$-\frac{1}{\xi(1+\xi^2)}UW - \frac{1}{2\eta}VW$	$\frac{2\xi^2-1}{\xi(1+\xi^2)}U - \frac{1}{2\eta}V - \frac{2\alpha}{\xi}\sqrt{\frac{1+\xi^2}{\eta}}W$
2.2	$-\alpha/\xi$	$-U + (1/\xi)W^2$	$-\beta - V - \frac{\alpha}{\xi}W + 2\frac{\alpha}{\xi^2}UW$	$-W - (1/\xi)UW$	$-3 - (1/\xi)U$
2.3	0	$(1-U)\left(\frac{1}{\xi} + \frac{2}{\eta}V\right)$	$-\frac{1}{\xi}V - \frac{1}{\xi}UV + \frac{\eta}{\xi^2\alpha^2}(1-U)$	$-(1/\xi)W$	$-3/\xi - (1/\eta)V$
2.4	0	$\frac{\xi}{\alpha^2} - U + \xi\left(\frac{2}{\alpha} + V\right)V$	$-\left(2\frac{U}{\xi} + 1\right)\left(V + \frac{1}{\alpha}\right)$	$-\beta/\alpha - \beta V$	$-(2 + U/\xi)$
2.5	0	0	$(1/\eta)W^2$	$-(1/\eta)VW$	$-(1/\eta)V$
2.6	$-1/\xi$	$(1/\xi)W^2$	$(2/\xi^2)UW$	$-(1/\xi)UW$	$-(1/\xi)U$
2.7	$-\alpha/\xi$	$(1/\xi)W^2$	$(2\alpha/\xi^2)UW - 1$	$-(1/\xi)UW$	$-(1/\xi)U$
2.11	0	$-(2/\eta)V(U+1)$	$\eta(U+1)^2$	$-\beta$	$-(1/\eta)V$
2.12	-2η	$-\frac{1}{\eta}UV + \frac{2\xi}{1+\xi^2}U^2 - UW$	$\frac{2\eta}{(1+\xi^2)^2}U^2 + \frac{1}{2\eta}V^2 - VW$	0	$\frac{2\xi}{1+\xi^2}U - 2W$
2.13	0	$-U$	$-V$	$-W$	-3
2.14	0	$-(U+\alpha)$	$-V$	$-W$	-3
2.15	0	$-(U+\alpha)$	$-V$	$-\beta$	-2
2.16	0	$-U$	$-V$	$-\beta$	-2
2.17	0	0	0	0	0
2.18	0	-1	0	$-\alpha$	0
2.19	0	0	0	-1	0

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